# Markowitz Portfolio Optimization

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## Abstract

Modern portfolio theory was pioneered by Markowitz in 1952 and led to him being awarded the Nobel Prize in Economics in 1990. The original essay on portfolio selection [7] has since inspired a multitude of researchers and analysts to develop theories on financial modelling and risk management. Seeking similar inspiration for future work, this report aims to study the classical portfolio optimization technique introduced by Markowitz and to apply it to real world data.

## 1 Portfolio Selection Problem

Borrowing the description from Boyd & Vandenberghe's textbook [1], we consider the classical portfolio problem with $n$ assets or stocks held over a period of time. Let $w_i$ denote the amount of asset $i$ held throughout the period, with $w_i$ in dollars, at the price at the beginning of the period. A normal long position in asset $i$ corresponds to $w_i > 0$; a short position in asset $i$ (*i.e.*, the obligation to buy the asset at the end of the period) corresponds to $w_i < 0$. We let $p_i$ denote the relative price change of asset $i$ over the period, *i.e.*, its change in price over the period divided by its price at the beginning of the period. The overall return on the portfolio is $r = p^T w$ (given in dollars). The optimization variable is the portfolio vector $w \in \mathrm{R}^n$.

A wide variety of constraints on the portfolio can be considered. The simplest set of constraints is that $w_i \geq 0$ (*i.e.*, no short positions) and $1^T w = B$ (*i.e.*, the total budget to be invested is $B$, which is often taken to be one).

We take a stochastic model for price changes: $p \in \mathrm{R}^n$ is a random vector, with known mean $\mu$ and covariance $\Sigma$. Therefore with portfolio $w \in \mathrm{R}^n$, the return $r$ is a (scalar) random variable with mean $\mu^T w$ and variance $w^T \Sigma w$. The choice of portfolio $w$ involves a trade-off between the mean of the return, and its variance.

The classical portfolio optimization problem is the quadratic program

$$\begin{array}{ll} \text{minimize} & w^T \Sigma w \\ \text{subject to} & \mu^T w \geq r_{min} \\ & 1^T w = 1, \quad w \succeq 0, \end{array}$$

where $w$, the portfolio, is the variable. Here we find the portfolio that minimizes the return variance (which is associated with the risk of the portfolio) subject to achieving a minimum acceptable mean return $r_{min}$, and satisfying the portfolio budget and no-shorting constraints.

Many variations of the original problem constraints are possible and are solved in Section 3 with synthetically generated data. In Section 4, we attempt to solve the problem for real world data. Our solutions are built with CVXPY, a Python-embedded modeling language for convex optimization problems.

# 2 Convexity Analysis

A convex optimization problem is the optimization of a convex function over a convex set. Such problems and the concept of convexity itself are an important topic in economics and finance- the behaviour of a rational agent can be modelled as maximizing an objective function, called the utility function, under various constraints. Henrotte & Lebret [3] described risk aversion as the maximization of a concave utility function, *i.e.* the willingness to pay to avoid a fair game. Hence, if we can model optimization problems in Economics and Finance as generic convex optimization problems, we can use various well developed techniques to solve these problems.

The quadratic optimization problem or quadratic program (QP) is a convex optimization problem where the objective function is (convex) quadratic and the constraint functions are affine. A typical QP can be expressed in the form

$$\begin{array}{ll} \text{minimize} & (1/2)\, x^T P x + q^T x + r \\ \text{subject to} & G\, x \preceq h \\ & A\, x = b, \end{array}$$

where $P$ is a symmetric positive semidefinite matrix, $G \in \mathbf{R}^{m \times n}$, and $A \in \mathbf{R}^{p \times n}$.

The portfolio selection problem described in Section 1 can be solved using techniques to solve a generic QP as long as the covariance matrix $\Sigma$ is symmetric positive semidefinite, since the objective is (convex) quadratic and the constraints are affine.

# 3 Synthetic Examples

Our goal is to demonstrate how to do portfolio optimization with CVXPY on synthetic data. We first generate the values of the mean price change $\mu$ and the covariance $\Sigma$ over an arbitrary period for $n$ assets (we take $n = 10$) using a randomizer. We ensure that $\Sigma$ is symmetric positive semidefinite to obtain a convex objective function.

In the following sections, we describe the experiments we conducted for several variants of the classical Markowitz problem. We made use of the code provided in [2] as a starting point. The appendix shows the values of $\mu$ and $\Sigma$ used for all experiments.

## 3.1 Using a Risk Aversion Parameter

By introducing a new parameter $\gamma > 0$ (called the *risk aversion parameter*) the classical portfolio optimization problem described in Section 1 can be reformulated as

$$\begin{array}{ll} \text{maximize} & \mu^T w - \gamma w^T \Sigma w \\ \text{subject to} & 1^T w = 1, \quad w \succeq 0, \end{array}$$

The objective $\mu^T w - \gamma w^T \Sigma w$ is the *risk-adjusted return*. Varying $\gamma$ gives the optimal risk-return trade-off.

We experimented with 100 different values of $\gamma$ between $10^{-2}$ to $10^3$, spaced evenly on a log scale. We solved the corresponding 100 QPs using CVXPY and plotted the optimal risk-return trade-off for the 10 assets in Figure 1. We also selected two values of $\gamma$ ($\gamma = 0.29$ and $\gamma = 1.05$) on the trade-off curve and plotted their return distributions in Figure 2.

Figure 1: **Optimal risk-return trade-off for 10 assets.** The green curve shows the trade-off for different values of $\gamma$. The 10 red markers are the coordinates denoting the (standard deviation, return) pair for each asset.

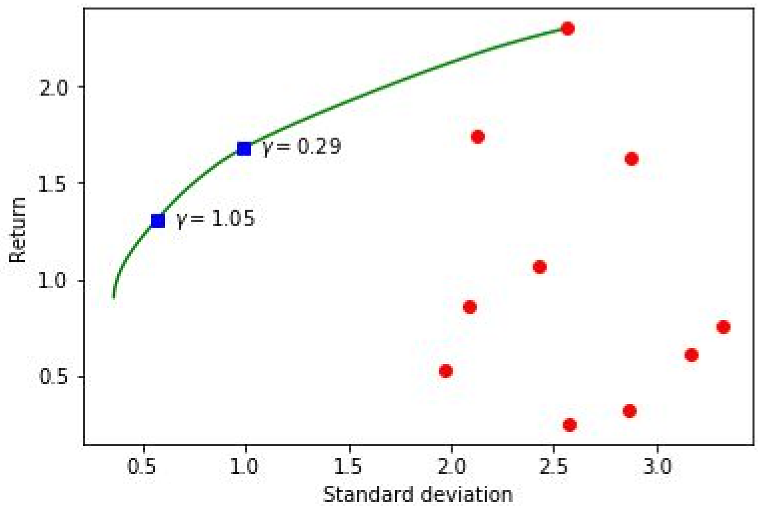
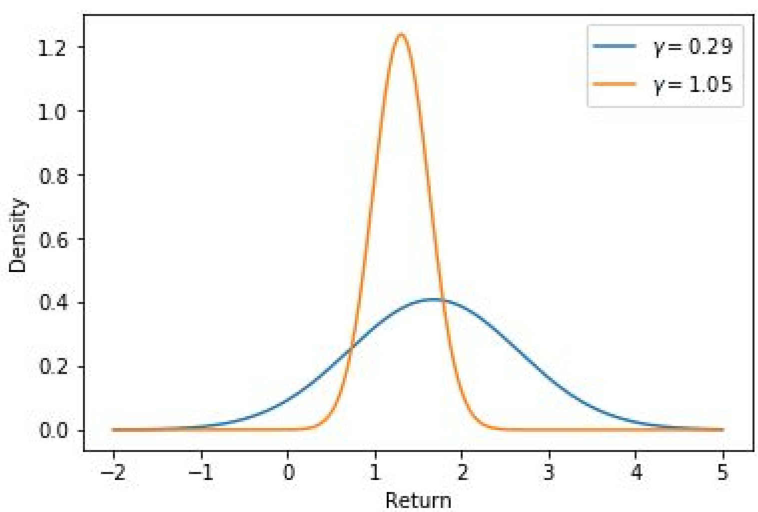


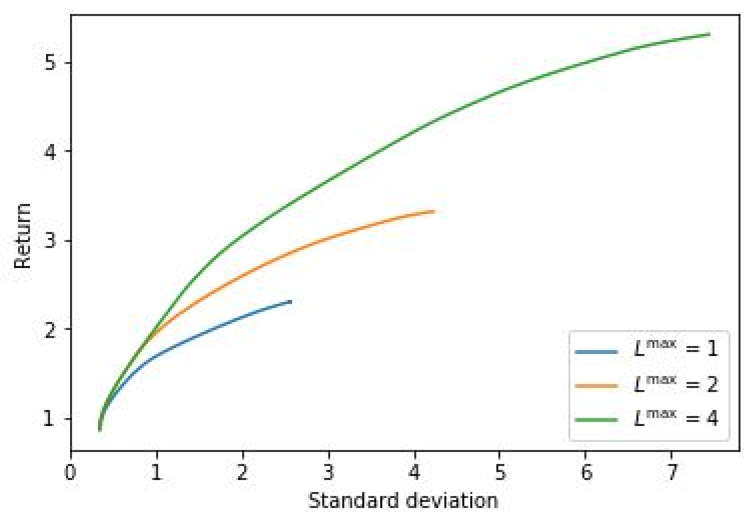

Figure 2: **Return distributions for two risk aversion values marked on the trade-off curve.** The probability of a loss is almost 0 for the low risk value and above 0 for the high risk value. (Higher $\gamma$ implies lower risk.)



## 3.2 Using Leverage Limits

Leverage is the investment strategy of using borrowed money, and can be modelled as the L1-norm of the allocation vector $w$, *i.e.* $\| w \|_1$. Building further on the problem defined in Section 3.1, we replaced the constraint of restricting ourselves to a long-only portfolio ($w$ □ $0$) with a new constraint on $w$ called the leverage limit, defined as $\| w \|_1 \leq L^{max}$. We solved the QP for leverage limits of 1, 2 and 4, and plotted the optimal risk-return trade-off curves in Figure 3.

Figure 3: **Trade-off curves for three values of $L^{max}$.** Allowing higher leverage leads to increased returns and allows greater risk.

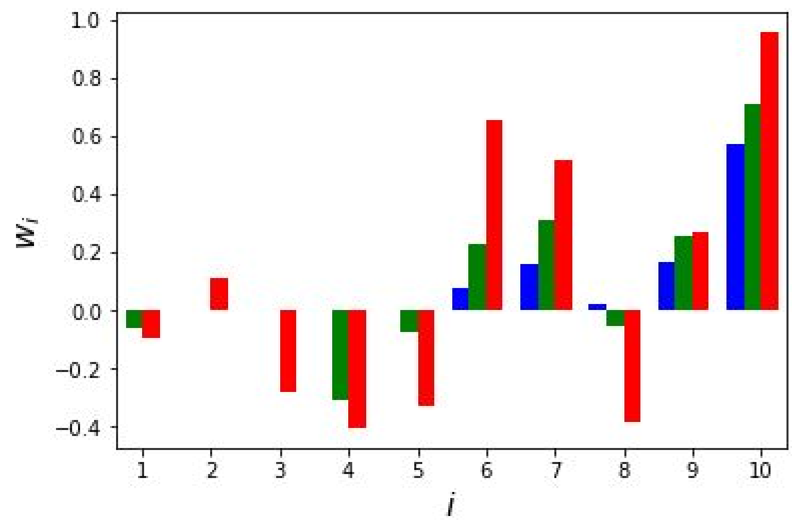



## 3.3 Bounding Risk

To analyze how a portfolio behaves when putting a bound on leverage, Figure 4 displays the amount of each asset held in each portfolio as a bar graph at the points on each trade-off curve where the risk $w^T \Sigma w = 2$. We solved the following QP

$$\begin{array}{ll} \text{maximize} & \mu^T w \\ \text{subject to} & 1^T w = 1 \\ & \| w \|_1 \leq L^{max} \\ & w^T \Sigma w \leq 2 \end{array}$$

Figure 4: **Amount of each asset held in each portfolio.** Note that blue, green and red bars correspond to $L^{max}$ taking the values 1, 2 and 4 respectively. We saw that higher leverage portfolios had a greater tendency to take up short positions (*i.e.* negative holdings) owing to higher tolerance for risk.



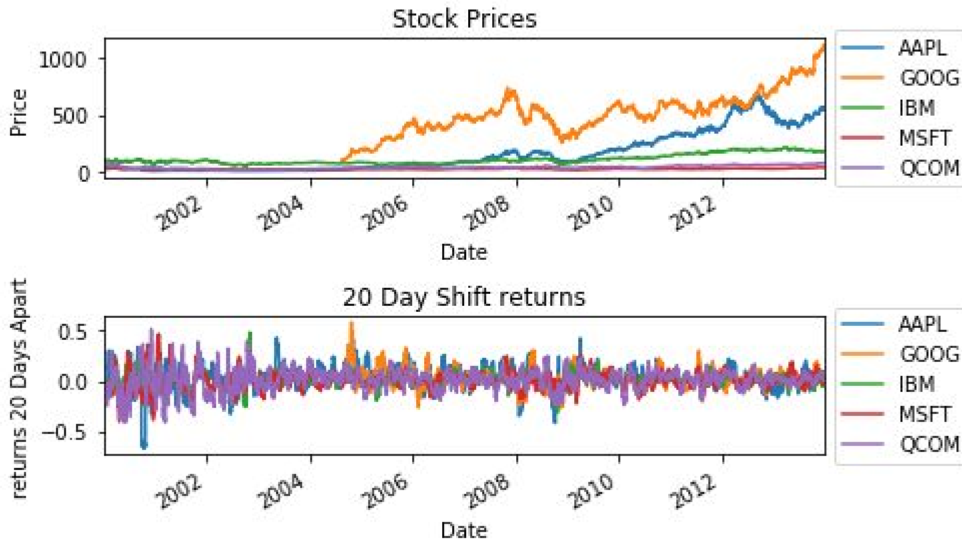

# 4 Real World Example

In this section, we apply the classical portfolio optimization technique on real world data. For our analysis, we selected five well known stocks (Apple, IBM, Google, Microsoft and Qualcomm) and a fixed interest rate as the six assets for which we want to optimize our portfolio allocation. We obtained the closing stock prices of the five stocks from January 2000 to January 2014 (Google's price is NaN till it was publically listed), and assigned a fixed value of 0.0025 to the interest rate (which we assume is a position with zero risk).

## 4.1 Estimating Mean Return and Covariance

To model the problem as a QP, we computed the mean return and covariance of the assets over a specific time period (we chose 20 days). The estimation of these parameters from historical prices is a popular topic in financial modelling, with many advanced techniques proposed for doing so. In our analysis, we chose a simple method where we first computed the shifted return of the assets over the time period specified as *shifted_return* = *price*[*current_date*]/*price*[*current_date* + *time_period*] - 1. Figure 5 shows the stock prices and shifted returns for each stock. We then used the exponentially weighted moving average, standard deviation and covariance functions provided by Pandas, a data analysis library for Python to compute the required parameters.

Figure 5: **Stock prices and shifted returns for the five stocks.**




However, the covariance matrix yielded by this method was not guaranteed to be symmetric positive semidefinite. To ensure problem feasibility, we replaced the covariance matrix by the nearest positive definite matrix using the technique proposed by Higham [4]. The computation of the nearest symmetric positive definite matrix is another well-studied topic in linear algebra [4] and financial modelling [5].

## 4.2 Solving the problem

Given the mean return $\mu$ and the covariance $\Sigma$, we solved the QP described in Section 1 to obtain the optimal portfolio allocation for six assets. Note that we restricted ourselves to a long-only portfolio with an explicit lower bound on return $r_{min} = 0.02$. We solved a series of QPs, starting from January 3rd, 2006 and re-optimizing every 20 days.

For each time period, we obtained the optimal portfolio allocation vectors. Figure 6 shows optimal distributions for the year 2013, while Figure 7 plots the portfolio return versus time. We used code from [6] to plot our results.

Figure 6: **Optimal portfolio allocation for the year 2013.**

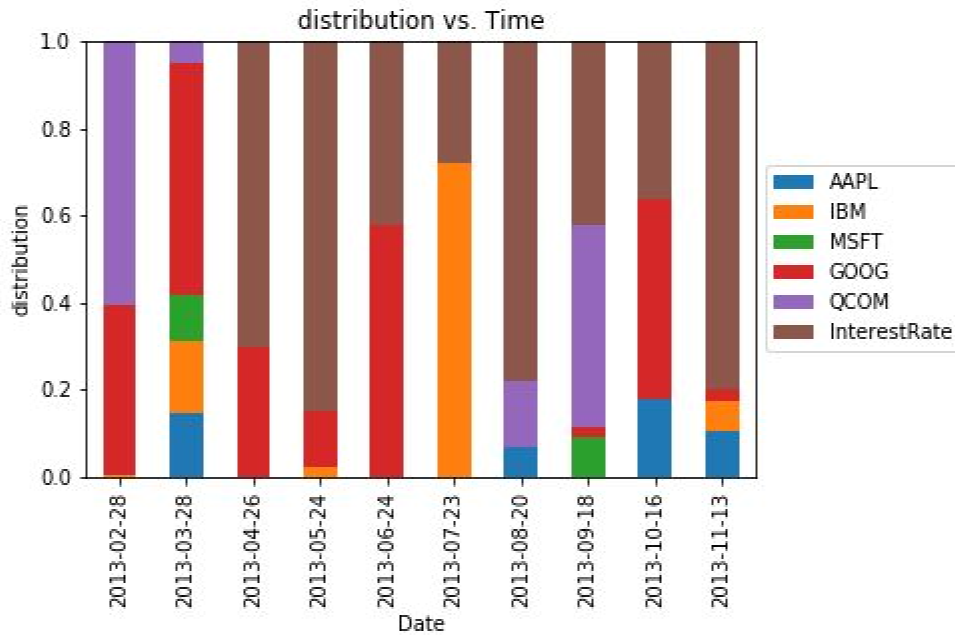
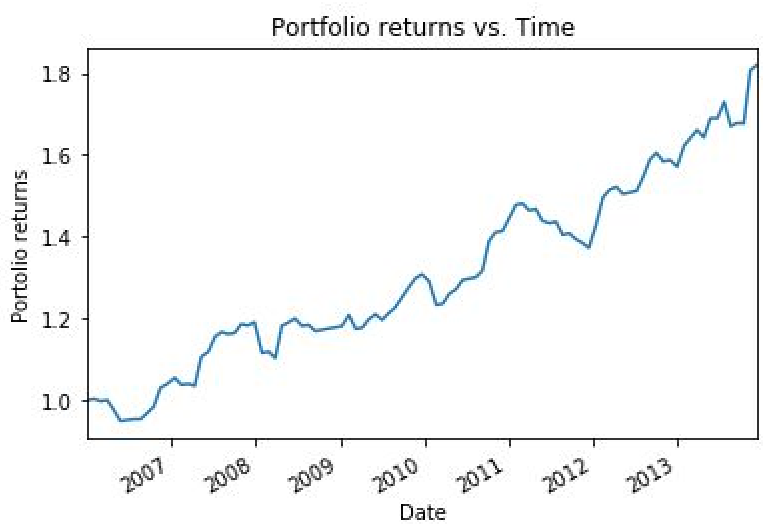



Figure 7: **Portfolio return versus time.** The approach yielded a portfolio that doubled in return in the specified time period. (However, our analysis was biased by deliberately picking stocks that have done well over the past years!)



# References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[2] S. Diamond and J. Park. Portfolio optimization examples (portfolio_optimization.ipynb), Materials for a short course on Convex Optimization. *GitHub repository,* 2016. Retrieved from *https://github.com/cvxgrp/cvx_short_course/*

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[4] N.J. Higham. Computing a nearest symmetric positive semidefinite matrix. *Linear Algebra and its Applications*. Vol. 103, pp. 103-118, 1988

[5] N.J. Higham. Computing the nearest correlation matrix - a problem from finance. *IMA Journal of Numerical Analysis.* Vol. 22, pp. 329-343, 2002

[6] M. Pugh. Code for plotting results (MarkowitzScript.py), Markowitz Portfolio Optimization. *Github repository*, 2014. Retrieved from *https://github.com/rexyroo/Markowitz/*

[7] H.M. Markowitz. Portfolio Selection. *Journal of Finance.* Vol. 7, pp. 77-91, 1952

# Appendix

Table 1: **Value for mean price change of 10 assets.**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1.62 | 0.61 | 0.53 | 1.07 | 0.87 | 2.30 | 1.74 | 0.76 | 0.32 | 0.25 |

Table 2: **Value of covariance matrix of 10 assets.**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 8.30 | -4.43 | -2.48 | -3.12 | 1.19 | 2.42 | -0.14 | -2.49 | -3.36 | 0.43 |
| -4.43 | 10.00 | 3.98 | 0.03 | -1.86 | 2.97 | 2.74 | -1.30 | 0.90 | -3.94 |
| -2.48 | 3.98 | 3.88 | 0.84 | 0.85 | -0.75 | 0.45 | -2.77 | -0.51 | -2.32 |
| -3.12 | 0.03 | 0.84 | 5.89 | 0.04 | -2.84 | 0.76 | 2.73 | -1.61 | 1.27 |
| 1.19 | -1.86 | 0.85 | 0.04 | 4.35 | -2.81 | -0.36 | 0.06 | -0.17 | 0.53 |
| 2.42 | 2.97 | -0.75 | -2.84 | -2.81 | 6.57 | 1.21 | -1.09 | -0.50 | -2.13 |
| -0.14 | 2.74 | 0.45 | 0.76 | -0.36 | 1.21 | 4.52 | 1.68 | 0.77 | -2.92 |
| -2.49 | -1.30 | -2.77 | 2.73 | 0.06 | -1.09 | 1.68 | 11.00 | 4.91 | 0.22 |
| -3.36 | 0.90 | -0.51 | -1.61 | -0.17 | -0.50 | 0.77 | 4.91 | 8.22 | -1.74 |
| 0.43 | -3.94 | -2.32 | 1.27 | 0.53 | -2.13 | -2.92 | 0.22 | -1.74 | 6.63 |